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"Second-Order Effects in High-Frequency Transionospheric Propagation"

by

W. J. Ross

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Submitted by: W. J. Ross

W. J. Ross, Professor of Electrical Engineering

Approved by: A. H. Waynick

A. H. Waynick, Professor of Electrical Engineering, Director, Ionosphere Research Laboratory

The Pennsylvania State University

College of Engineering

Department of Electrical Engineering

TABLE OF CONTENTS

1. Introduction

In recent years the propagation of high frequency radio waves through the ionosphere has formed the basis of a number of experimental techniques for the study of the ionosphere. By measuring the resultant properties of the received signals from, for example, lunar reflections of radar pulses, or from rocket or satellite beacon transmitters, it has been possible to deduce many properties of the ionospheric medium, including that part above the level of maximum ionization density which had not previously been readily accessible to radio wave probing.

Consideration will here be limited to waves much higher than both the plasma frequency and the electron gyro-frequency, and on which, therefore, the effect of the medium is relatively small. For such waves the effect of the ionospheric medium is largely one of refraction and birefringence and attention is directed to the resulting phase and polarization rotation effects produced in the waves.

Both these effects are cumulative over the propagation path and their measurement leads quite directly to an approximate measure of the total quantity of ionization along the propagation path.

If the usual "high-frequency approximation" is made that all rays may be considered to travel the same straight line path from source to receiver, then the relative refractive index may be written approximately

$$\mu = 1 - \frac{1}{2} X \pm \frac{1}{2} XY_L$$

where $X = \dfrac{Ne^2}{4\pi^2 \varepsilon_o mf^2}$

$$Y_L = \frac{e\,B_L}{2\pi m f}$$

N = the electron density

e = the electronic charge

ε_o = the electrical permittivity of free space

m = the electronic mass

f = the wave frequency

B_L = the component of the magnetic field along the straight line

Then to a first order the polarization rotation Ω_o may be found by integrating the differential refractive index between the two magneto-ionic modes along the straight line, giving

$$\Omega_o = \frac{\pi}{\lambda} \int_o^h X Y_L \sec\theta \, dh \quad \text{radians}$$

$$= \frac{\pi}{\lambda} \overline{Y_L \sec\theta} \int X \, dh \qquad (1)$$

where λ is the free space wavelength

θ is the zenith angle at height h

$\overline{Y_L \sec\theta}$ is the weighted mean value of $Y_L \sec\theta$ along the straight line.

and $\int X \, dh$ is proportional to the integrated vertical column of electrons over the height range of the measurement.

Similarly the reduction ΔP_o in the phase path relative to that of a free space medium is given by

$$\Delta P_o = \frac{1}{2} \overline{\sec\theta} \int X \, dh \qquad (2)$$

where $\overline{\sec\theta}$ is the weighted mean value of $\sec\theta$.

The assumption of straight line propagation is equivalent to the medium being of uniform ionization density over the entire propagation path and having a uniform magnetic field. In addition, the linear dependence of the wave perturbations on the electron integral is based on the assumption that the refractive indices for the modes considered are linear in electron density, a condition which is approached asymptotically at high frequencies.

It is the purpose of this paper to relax the above simplifying approximations and to develop propagation equations which will describe these effects to a higher order of accuracy in a form suitable for manual data reduction.

The ways in which propagation through the ionosphere departs from the description of first order theory have been considered previously by a number of workers. Their inclusion in the analysis of data has, however, been done fully only through the use of digital computing programs which seek solutions to the propagation equations in terms of ionospheric models whose parameters are adjusted numerically to optimize the fit to the data, e.g., Garriott (1960), Lawrence and Posakony (1961).

2. Second-Order Effects

The departures from the straight line approximation for waves propagating through the ionosphere may be listed as follows:

(1) The non-uniform distribution of ionization causes the various rays to be refracted and follow different paths between source and receiver.

(2) Since the medium is anisotropic the wavenormal and ray for a particular mode of propagation are not coincident in direction.

(3) The refractive index is non-linear in electron density and magnetic field intensity.

It would be most realistic to formulate the propagation equations in terms of a spherically stratified ionosphere model, with a magnetic field of approximately dipole form. However, the complexity of this system for purposes of analysis is very great, and we have here resorted to a more approximate model, in order to make the analysis more tractable. We shall replace the spherically stratified model by a plane stratified model in which the originally spherical ionization contours are replaced by a set of parallel plane contours tangent to the former at the point where the straight line path intersects some central height of the ionization distribution. For convenience, we shall assume that this point is the same as that at which the mean longitudinal field component is evaluated for the first order theory in Equation (1). In addition, the non-uniform magnetic field is replaced by a uniform field having the magnitude and direction of the original field at the same "ionospheric point".

The most important simplification resulting from this model is that the wavenormal associated with the ray in any uniform lamination of the medium does not change its direction relative to the plane of stratification as a result of ray-wavenormal angular separation, and therefore it is possible to calculate the refraction of the wavenormal in terms of a simple Snell's Law expression. The methods of ray-optics will be used throughout.

We shall seek solutions to the ray paths through this model and consider that the perturbations to the first order theory which are found are a close approximation to the perturbations which would be present in the more sophisticated model.

Since it is often convenient, or even necessary, to compute magnetic field maps in a form which may be used to reduce all data, these computations must be made with respect to some known frame of reference, most conveniently taken to be the straight line joining source and receiver which is used for first order theory computations. We shall therefore refer all directions and quantities to this straight line.

3. The Propagation Equations

We shall assume the relative wave refractive index μ of the medium to be purely real and given by the collision-free, quasi-longitudinal form of the Appleton-Hartree equation

$$\mu^2 \doteqdot 1 - \frac{X}{1 \pm Y \cos\psi - Y^2 \sin^2\psi/2\,(1 - X)} \tag{3}$$

where ψ is the angle between the field and the wavenormal. We shall denote the direction of the ray associated with a wavenormal unit vector $\underset{\sim}{w}$ by a unit vector $\underset{\sim}{r}$.

Then, through the anisotropic optics which apply in this situation, e.g. Budden (1961), the vectors $\underset{\sim}{w}$, $\underset{\sim}{r}$ and $\underset{\sim}{B}$ are coplanar, with $\underset{\sim}{r}$ rotated from $\underset{\sim}{w}$ toward $\underset{\sim}{B}$ by an angle whose tangent is $\frac{1}{\mu}\,\frac{\partial\mu}{\partial\psi}$. Denoting this ray-wavenormal separation angle by α , we may write alternatively

$$\tan\alpha = \frac{1}{2\mu^2}\,\frac{\partial\mu^2}{\partial\psi}$$

$$= \mp\,\frac{XY\sin\psi\,[1 \pm Y\cos\psi/(1 - X)]}{2\,\mu^2\,[1 \pm Y\cos\psi - Y^2\sin\psi/2\,(1 - X)]^2} \tag{4}$$

Thus we may write $\underset{\sim}{r} = \ell\underset{\sim}{w} + m\underset{\sim}{Y}$ (5)

where

$$m = \frac{\sin\alpha}{Y\sin\psi} \tag{6}$$

and

$$\ell = \frac{\sin(\psi - \alpha)}{\sin\psi} \tag{7}$$

$$= \cos\alpha - \cot\psi\sin\alpha$$

Now establish a rectangular Cartesian coordinate system at the "ionospheric point" on the straight line from source to receiver, with the z-direction vertically upwards and the x-direction horizontal in the vertical plane of the straight line. In this frame the wavenormal associated with a given ray will have a constant azimuthal direction ϕ_o.
Denote the azimuthal angle for the ray by ϕ.
Denote the zenith angle of the wavenormal by i, of the ray by χ, and of the straight line path by θ. The geometry of this system is shown schematically in Figure 1. Then the refraction of the wavenormal may be calculated from Snell's Law

$$\mu \sin i = \sin i_o \tag{8}$$

where i_o is the angle of incidence of the wavenormal (and of the ray) at the receiver end of the propagation path, at which the ionization density is assumed to be zero.

The phase path length P of the ray from source to receiver is then found by integrating the ray velocity along the ray path.

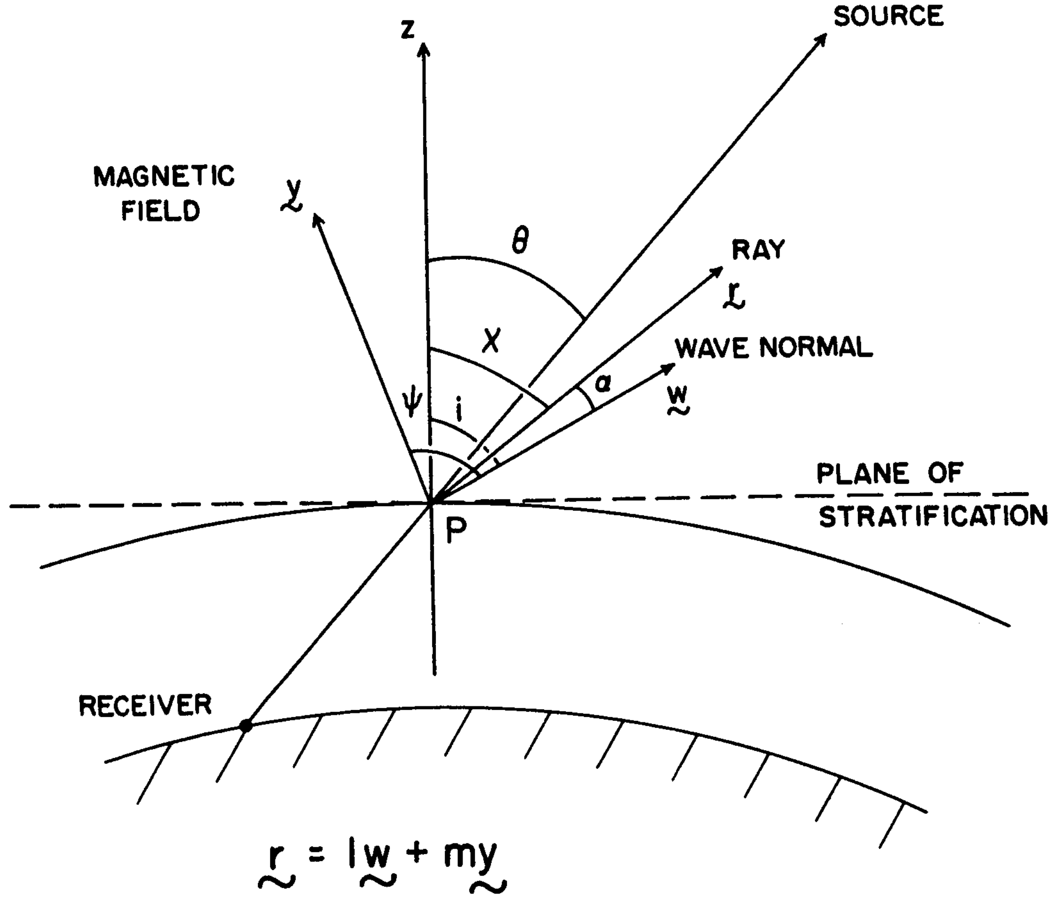
$$P = \int_h \mu \cos\alpha \quad \sec\chi \ dz \tag{9}$$

The constraint that the ray should have end points at source and receiver may be expressed by the two integral equations

$$\int_h r_x \sec \chi \ dz = h \tan \theta \tag{10}$$

and

$$\int_h r_y \sec \chi \ dz = 0 \tag{11}$$



GEOMETRICAL RELATIONS AT THE IONOSPHERE POINT, P

FIGURE I

where each of the integrals in Equations (9), (10) and (11) is taken over h, the vertical separation of source and receiver.

The desired solution of the above set of Equations (3) - (11) is to express P in terms of the known quantities θ, h, X and Y. An exact solution is not readily arrived at, due largely to the two integral equations of the set, and it is necessary to approximate the problem further. We shall expand all equations as polynomial series in X and Y (or its components) and shall seek a solution by retaining all terms up to order three in these quantities. It should be noted that the first order polarization rotation equation (Equation (1)) is of order two so that the above procedure should add an additional level of accuracy to the result.

4. Solution of the Equations

We see from Equation (4) that α is of order two in its leading term, and therefore we may write $\alpha \doteqdot \tan\alpha \doteqdot \sin\alpha$.

$$\cos\alpha \doteqdot 1$$

Also since the angles ϕ_o and ϕ arise directly from the azimuthal components of α, similar approximations apply to their trigonometric functions.

In order to abbreviate the equations we shall write

$$\mu^2 = 1 - A$$

where

$$A = \frac{X}{1 \pm Y\cos\psi - Y^2\sin^2\psi/2(1-X)}$$

$$\doteqdot X[1 \mp Y\cos\psi + Y^2\cos^2\psi + \tfrac{1}{2}Y^2\sin^2\psi] \tag{12}$$

Thus A is of order unity in its leading term.

Then
$$\sin^2 i = \frac{\sin^2 i_o}{1 - A}$$

$$\doteqdot \sin^2 i_o\,[1 + A + A^2 + A^3]$$

$$\therefore \cos^2 i = \cos^2 i_o\,[1 - \tan^2 i_o\,(A + A^2 + A^3)] \tag{13}$$

From Equation (5)

$$\cos\chi = r_z = \ell\cos i + mY_z$$

$$= \cos i\,[1 + B] \tag{14}$$

where

$$B \doteqdot \frac{\alpha}{\sin \psi} \left[\frac{Y_z}{Y} \sec i - \cos \psi \right] \tag{15}$$

Thus B is of order two in its leading term.

$$\text{Then } \cos^2 \chi \doteqdot \cos^2 i \; [\, 1 + 2B \,]$$

$$= \cos^2 i_o \; [\, 1 - A \tan^2 i_o - A^2 \tan^2 i_o - A^3 \tan^2 i_o$$

$$+ 2B - 2AB \tan^2 i_o \,]$$

Inverting

$$\sec^2 \chi = \sec^2 i_o \, [\, 1 + A \tan^2 i_o + A^2 \tan^2 i_o \sec^2 i_o + A^3 \tan^2 i_o \sec^4 i_o - 2B]$$

$$\therefore \tan^2 \chi = \sec^2 \chi - 1$$

$$= \tan^2 i_o \, [\, 1 + A \sec^2 i_o + A^2 \sec^4 i_o + A^3 \sec^6 i_o$$

$$- 2B \operatorname{cosec}^2 i_o \,]$$

Taking the square root of both sides gives

$$\tan \chi \doteqdot \tan i_o \; [\, 1 + \tfrac{1}{2} A \sec^2 i_o + \tfrac{3}{8} A^2 \sec^4 i_o + \tfrac{5}{16} A^3 \sec^6 i_o - B \operatorname{cosec}^2 i_o$$

$$+ \tfrac{1}{2} AB \sec^2 i_o \operatorname{cosec}^2 i_o \,] \tag{16}$$

The constraint of Equation (10) can be written

$$h \tan \theta = \int_h \tan \chi \cos \phi \, dz$$

$$\doteqdot \int_h \tan \chi \, dz \tag{17}$$

Therefore integrating both sides of Equation (16) over the height range h gives

$$\tan\theta = \tan i_o \left[1 + \frac{1}{2}\,\overline{A}\,\sec^2 i_o + \frac{3}{8}\,\overline{A^2}\,\sec^4 i_o + \frac{5}{16}\,\overline{A^3}\,\sec^6 i_o - \overline{B}\,\operatorname{cosec}^2 i_o + \frac{1}{2}\,\overline{AB}\,\sec^2 i_o \operatorname{cosec}^2 i_o \right] \qquad (18)$$

where the bars denote height averages over the range of integration. Inverting (18)

$$\tan i_o = \tan\theta \left[1 - \frac{1}{2}\,\overline{A}\,\sec^2 i_o - \frac{3}{8}\,\overline{A^2}\sec^4 i_o - \frac{5}{16}\,\overline{A^3}\sec^6 i_o + \overline{B}\,\operatorname{cosec}^2 i_o - \frac{1}{2}\,\overline{AB}\,\sec^2 i_o \operatorname{cosec}^2 i_o + \frac{1}{4}\,\overline{A}^2\sec^4 i_o + \frac{3}{8}\,\overline{A}\,\overline{A^2}\,\sec^6 i_o - \overline{A}\,\overline{B}\,\sec^2 i_o \operatorname{cosec}^2 i_o - \frac{1}{8}\,\overline{A}^3\sec^6 i_o \right] \qquad (19)$$

Equation (19) forms the basis of calculating trigonometric functions of the angle i_o in terms of θ, to any desired order of accuracy up to three, by iteration. Since these functions will most often be used in squared form it is convenient to square Equation (19) before evaluating the functions of i_o.

$$\tan^2 i_o = \tan^2\theta \left[1 - \overline{A}\,\sec^2 i_o - \frac{3}{4}\,\overline{A^2}\sec^4 i_o - \frac{5}{8}\overline{A^3}\sec^6 i_o + 2\overline{B}\operatorname{cosec}^2 i_o - \overline{AB}\,\sec^2 i_o \operatorname{cosec}^2 i_o + \frac{3}{4}\,\overline{A}^2\,\sec^4 i_o + \frac{9}{8}\,\overline{A}\,\overline{A^2}\,\sec^6 i_o - 3\,\overline{A}\,\overline{B}\,\sec^2 i_o \operatorname{cosec}^2 i_o - \frac{1}{2}\,\overline{A}^3\,\sec^6 i_o \right] \qquad (20)$$

To order zero

$$\tan^2 i_o = \tan^2 \theta \tag{21}$$

To order unity

$$\tan^2 i_o = \tan^2 \theta \,[\, 1 - \bar{A} \sec^2 \theta] \tag{22a}$$

$$\therefore \sec^2 i_o = \sec^2 \theta \,[\, 1 - \bar{A} \tan^2 \theta] \tag{22b}$$

$$\sec^4 i_o = \sec^4 \theta \,[\, 1 - 2\bar{A} \tan^2 \theta] \tag{22c}$$

$$\mathrm{cosec}^2\, i_o = \mathrm{cosec}^2\, \theta \,[\, 1 + \bar{A} \,] \tag{22d}$$

To order two

$$\tan^2 i_o = \tan^2 \theta \,[\, 1 - \bar{A} \sec^2 \theta + \bar{A}^2 \tan^2 \theta \sec^2 \theta + 2\bar{B} \,\mathrm{cosec}^2\, \theta + \tfrac{3}{4} (\bar{A}^2 - \overline{A^2}) \sec^4 \theta \,] \tag{23a}$$

$$\therefore \sec^2 i_o = \sec^2 \theta \,[\, 1 - \bar{A} \tan^2 \theta + \bar{A}^2 \tan^4 \theta + 2\bar{B} + \tfrac{3}{4} (\bar{A}^2 - \overline{A^2}) \tan^2 \theta \sec^2 \theta \,] \tag{23b}$$

To order three

$$\begin{aligned} \tan^2 i_o = \tan^2 \theta \,[\, 1 &- \bar{A} \sec^2 \theta \left\{ 1 - \bar{A} \tan^2 \theta + \bar{A}^2 \tan^4 \theta + 2\bar{B} \right. \\ &\left. + \tfrac{3}{4} (\bar{A}^2 - \overline{A^2}) \tan^2 \theta \sec^2 \theta \right\} + \tfrac{3}{4} (\bar{A}^2 - \overline{A^2}) \sec^4 \theta \left(1 - 2\bar{A} \tan^2 \theta \right) \\ &- \tfrac{5}{8} \overline{A^3} \sec^6 \theta + 2\bar{B} \,\mathrm{cosec}^2\, \theta \,(1 + \bar{A}) - \overline{AB} \sec^2 \theta \,\mathrm{cosec}^2\, \theta \\ &+ \tfrac{9}{8} \bar{A}\, \overline{A^2} \sec^6 \theta - 3 \bar{A} \bar{B} \sec^2 \theta \,\mathrm{cosec}^2\, \theta - \tfrac{1}{2} \bar{A}^3 \sec^6 \theta \,] \end{aligned} \tag{24a}$$

$$\therefore \quad \sec^2 i_o = \sec^2\theta\,[\,1 - \bar{A}\tan^2\theta \left\{ 1 - \bar{A}\tan^2\theta + \bar{A}^2\tan^4\theta + 2\bar{B} \right.$$

$$\left. + \frac{3}{4}(\bar{A}^2 - \overline{A^2})\tan^2\theta\sec^2\theta \right\} + \frac{3}{4}(\bar{A}^2 - \overline{A^2})\tan^2\theta\sec^2\theta\,(1 - 2\bar{A}\tan^2\theta)$$

$$- \frac{5}{8}\,\overline{A^3}\sec^4\theta\tan^2\theta + 2\bar{B}\,(1 + \bar{A}) - \overline{AB}\sec^2\theta + \frac{9}{8}\bar{A}\,\overline{A^2}\sec^4\theta\tan^2\theta$$

$$+ 3\,\bar{A}\,\bar{B}\sec^2\theta - \frac{1}{2}\bar{A}^3\sec^4\theta\tan^2\theta\,] \qquad (24b)$$

We now have the means of evaluating the integrand of Equation (9)

$$\mu^2\sec^2\chi\,\cos^2\alpha \doteq \mu^2\sec^2\chi$$

$$= (1 - A)\sec^2 i_o\,[\,1 + A\tan^2 i_o + A^2\tan^2 i_o\sec^2 i_o$$

$$+ A^3\tan^2 i_o\sec^4 i_o - 2B\,]$$

$$= \sec^2 i_o\,[\,1 - A + A\tan^2 i_o + A^2\tan^4 i_o + A^3\tan^4 i_o\sec^2 i_o$$

$$- 2B + 2AB\,] \qquad (25)$$

It is now a straight forward though lengthy procedure to substitute in Equation (25) for the functions of i_o from the appropriate set (21) through (25) and to insert appropriate expressions for A and B evaluated in terms of θ also.

From Equation (12)

$$A = X\,[\,1 \mp Y\cos\psi + Y^2\cos^2\psi + \frac{1}{2}Y^2\sin^2\psi\,]$$

But $Y\cos\psi = Y_x\sin i\cos\phi_o + Y_y\sin i\sin\phi_o + Y_z\cos i$

$$\doteq Y_x\sin i + Y_z\cos i$$

$$\doteq Y_L - \frac{1}{2}(X - \bar{X})\,Y_1\tan\theta \qquad (26)$$

and $\quad Y^2\sin^2\psi \doteq Y^2 - Y_L^2$

Where Y_L is the component of Y along the straight line path, measured positively upwards,

and Y_1 is the component of Y perpendicular to the straight line path, in the plane of incidence, and measured positively upwards.

$$\therefore A = X\left[1 \mp Y_L \pm \frac{1}{2}(X - \overline{X})\, Y_1 \tan\theta + \frac{1}{2} Y_L^{2} + \frac{1}{2} Y^{2}\right] \qquad (27)$$

From Equations (15) and (4)

$$B \doteqdot \mp \frac{X\,(Y_z \sec i - Y \cos\psi)}{2\,(1 - X)(1 \mp Y_L)}$$

which reduces to

$$B = \mp \frac{1}{2} X Y_1 \tan\theta \left[1 + X \mp Y_L + \frac{1}{2}(X - \overline{X})\left(\sec^{2}\theta + \frac{Y_L}{Y_1}\tan\theta\right)\right] \qquad (28)$$

We shall not attempt to evaluate the phase path integral of Equation (9) to order three by this direct approach, but shall consider separately the cases relevant to the two phenomena of interest, namely polarization rotation and phase path dispersion, for which more straightforward but less direct methods will be used. Before proceeding, however, it is of interest to note certain properties of the equations.

(1) In the final equations above, the component of Y normal to the plane of incidence appears only once, in the final term of Equation (27), and in the same form for each magnetoionic mode. Its only effect is to modify the effective value of X slightly in all the refraction effects evaluated to order three. It is therefore evident that the effects of refraction out of the plane of incidence are at most very minor in importance, and will be seen later to be negligible to this order of accuracy.

(2) All angle functions appearing in the expression for wave refractive index are multiplied by coefficients of order two or greater, and therefore the angles need only be evaluated to order unity, i.e., to the accuracy of the no-field ray, in order to compute the refractive indices. However, the ray direction still needs to be evaluated to order three in order to find the path length to this accuracy.

4.1 Polarization Rotation

For the consideration of second-order effects in the polarization rotation of a wave propagating through the ionosphere, the quantity of interest is the phase path difference between the two circularly polarized magneto-ionic modes which are represented in the equations by the alternate signs. Denoting the difference between these modes by the operator Δ, we note

$$\Delta(\mu \sec \chi) = \frac{\Delta(\mu^2 \sec^2 \chi)}{\mu_1 \sec \chi_1 + \mu_2 \sec \chi_2} \tag{29}$$

and since the numerator of this expression is of order two in its leading term, the denominator need be evaluated to terms of order unity only. From Equation (25)

$$\begin{aligned}\Delta(\mu^2 \sec^2 \chi) = {} & \Delta(\sec^2 i_o) - \Delta(A \sec^2 i_o) + \Delta(A \tan^2 i_o \sec^2 i_o) \\ & + \Delta(A^2 \tan^4 i_o \sec^2 i_o) + \Delta(A^3 \tan^4 i_o \sec^4 i_o) - 2\Delta(B \sec^2 i_o) \\ & + 2\Delta(AB \sec^2 i_o)\end{aligned}$$

On substituting for i_o to the appropriate degree of accuracy and evaluating these terms, and noting that the difference operation on any triple product of A and $\overline{A}$ is identically zero, we find

$$\Delta(\mu^2 \sec^2 \chi) = \sec^2\theta\,[-\Delta(A) + \tan^2\theta\,\Delta(A - \overline{A}) + \tan^4\theta\,\Delta(A - \overline{A})^2$$

$$- \frac{3}{4}\tan^2\theta\sec^2\theta\,\Delta(\overline{A^2} - \overline{A}^2) - 2\Delta(B - \overline{B})$$

$$+ \Delta(AB - \overline{A}\,\overline{B}) + \Delta(AB - \overline{AB}) + \tan^2\theta\,\Delta(A\overline{B} - \overline{AB})$$

$$+ 5\tan^2\theta\,\Delta(A\overline{B} - \overline{A}\,\overline{B})\,] \qquad (30)$$

From Equations (27) and (28)

$$\Delta(A) = -2XY_L + X(X - \overline{X})\,Y_1\tan\theta$$

$$\Delta(A^2) = -\,4X^2Y_L$$

$$\Delta(B) = -\,XY_1\tan\theta\,[\,1 + X + \frac{1}{2}(X - \overline{X})(\sec^2\theta + \frac{Y_L}{Y_1}\tan\theta)]$$

$$\Delta(AB) = -\,X^2Y_1\tan\theta$$

with similar expressions for barred and partially barred arguments. Substituting into Equation (30) gives

$$\Delta(\mu^2\sec^2\chi) = \sec^2\theta\,[2XY_L - XY_1(X - \overline{X})\tan\theta - 2Y_L(X - \overline{X})\tan^2\theta$$

$$+ Y_1(X^2 - \overline{X^2})\tan^3\theta - \overline{X}Y_1(X - \overline{X})\tan^3\theta$$

$$+ 2Y_1(X - \overline{X})\tan\theta + 2Y_1(X^2 - \overline{X^2})\tan\theta$$

$$+ Y_1(X^2 - \overline{X^2})\tan\theta\,(\sec^2\theta + \frac{Y_L}{Y_1}\tan\theta)$$

$$- Y_1\overline{X}(X - \overline{X})(\sec^2\theta + \frac{Y_L}{Y_1}\tan\theta) - 4Y_L(X - \overline{X})^2\tan^4\theta$$

$$+ 3Y_L(\overline{X^2} - \overline{X}^2)\tan^2\theta\sec^2\theta - 2\overline{X}^2\,Y_1\tan\theta + 2\overline{X}^2\,Y_1\tan^3\theta$$

$$+ \; 3\overline{X}^2 \, Y_1 \tan\theta \sec^2\theta - 6 \, X \, \overline{X} \, Y_1 \tan^3\theta$$

$$- \; 2 \, X^2 \, Y_1 \tan\theta + \overline{X^2} \;\; Y_1 \tan\theta \sec^2\theta \,] \qquad (31)$$

Now

$$\frac{1}{\mu_1 \sec \chi_1 + \mu_2 \sec \chi_2} = \frac{1}{2} \cos\theta \, [\, 1 + \frac{1}{2} X - \frac{1}{2} (X - \overline{X}) \tan^2\theta] \qquad (32)$$

On substituting Equations (31) and (32) into Equation (29), averaging over the height range of integration and simplifying

$$\overline{\Delta(\mu \sec \chi)} = \overline{X} \; Y_L \sec\theta \, [\, 1 + \frac{1}{2} \beta \overline{X} + \frac{1}{2} (\beta - 1) \overline{X} \tan\theta \, (\tan\theta - \frac{Y_1}{Y_L})]$$

$$= \overline{X} \;\; Y_L \sec\theta \, [\, 1 + \frac{1}{2} \beta \overline{X} + \frac{1}{2} (\beta - 1) \, G \, \overline{X} \,] \qquad (33)$$

where $\beta \overline{X}^2 = \overline{X^2}$ (34)

and $G = \tan\theta \, (\tan\theta - \frac{Y_1}{Y_L})$ (35)

Since the factor outside the bracket in Equation (33) is the result of first order straight line theory, we may write the polarization rotation angle, Ω, as determined by second-order theory, in terms of the rotation angle Ω_o given by first-order theory as

$$\Omega = \Omega_o \;\, [\, 1 + \frac{1}{2} \beta \, \overline{X} + \frac{1}{2} (\beta - 1) \, G \, \overline{X} \,] \qquad (36)$$

where

$$\Omega_o = \frac{\pi}{\lambda} \;\; h \, \overline{X} \;\; Y_L \;\; \sec\theta \qquad (37)$$

4.2 Phase Path Dispersion

The first order analysis of phase path reduction by refraction (Equation (2))shows that the result is of order unity in the leading term. To extend the accuracy of the result one additional order, it is necessary to evaluate Equation (9) to order two only. From Equation (25) the square of the integrand of Equation (9) can be written to this order as

$$\mu^2 \sec^2 \chi \cos^2 \alpha = \sec^2 i_o [1 - A + A \tan^2 i_o + A^2 \tan^4 i_o - 2B] \tag{38}$$

On substituting the appropriate values of the functions of i_o, and simplifying

$$\mu^2 \sec^2 \chi = \sec^2 \theta [1 - A + (A - \overline{A}) \tan^2 \theta - 2(B - \overline{B}) + (A - \overline{A})^2 \tan^4 \theta - \frac{3}{4} (\overline{A^2} - \overline{A}^2) \tan^2 \theta \sec^2 \theta]$$

Hence

$$\begin{aligned} \overline{\mu \sec \chi} &= \sec \theta \left[1 - \frac{1}{2} \overline{A} - \frac{1}{8} \overline{A^2} - \frac{1}{8} (\overline{A^2} - \overline{A}^2) \tan^2 \theta\right] \\ &= \sec \theta \left[1 - \frac{1}{2} \overline{X} \mp \frac{1}{2} \overline{X} Y_L - \frac{1}{8} \beta \overline{X}^2 - \frac{1}{8} (\beta - 1) \overline{X}^2 \tan^2 \theta\right] \end{aligned} \tag{39}$$

The reduction in phase path length from its free space value is then

$$h(\sec \theta - \overline{\mu \sec \chi}) = \frac{1}{2} h \overline{X} \sec \theta [1 \mp Y_L + \frac{1}{4} \beta \overline{X} + \frac{1}{4} (\beta - 1) \overline{X} \tan^2 \theta]$$

i.e. $$\Delta P = \Delta P_o [1 \mp Y_L + \frac{1}{4} \beta \overline{X} + \frac{1}{4} (\beta - 1) \overline{X} \tan^2 \theta] \tag{40}$$

where $$\Delta P_o = \frac{1}{2} h \overline{X} \sec \theta \tag{41}$$

= the first order theory value of phase path reduction.

5. Discussion

The second-order equations for polarization rotation, (Equation (36)), and phase path reduction, (Equation (40)), are each quadratic in $\overline{X}$, with coefficients involving known parameters of the straight line from source to receiver and of the magnetic field, and also the parameter β which may be estimated approximately. Rather than carry through a full quadratic solution for $\overline{X}$, it will probably usually be most convenient to first find an approximate solution from the first order Equations (37) and (41) respectively, and iterate once for a second-order solution. It is then most convenient to invert Equations (36) and (40) to give respectively

$$\overline{X} = \overline{X}' \left[1 - \frac{1}{2} \beta \overline{X}' - \frac{1}{2} (\beta - 1) G \overline{X}' \right] \qquad (42)$$

where $$\overline{X}' = \frac{\lambda}{\pi h Y_L \sec \theta} \cdot \Omega$$

and $$\overline{X} = \overline{X}' \left[1 \pm Y_L - \frac{1}{4} \beta \overline{X}' - \frac{1}{4} (\beta - 1) \overline{X}' \tan^2 \theta \right] \qquad (43)$$

where $$\overline{X}' = \frac{2 \Delta P}{h \sec \theta}$$

Further iteration is usually not justified in view of the approximate form of the equations and of their rapid convergence when $\overline{X}$ is small.

5.1 The Distribution Parameter, β

The parameter $\beta = \overline{X^2} / \overline{X}^2$ is a measure of the non-uniformity with which the ionization is distributed over the height of integration. It has a minimum value of unity for a completely uniform distribution, with larger values as the distribution is more concentrated. For a uniform slab layer occupying a fraction $\frac{1}{n}$ of the height range, β takes the value of n, while for a Chapman layer of the form $N = N_m \exp \frac{1}{2} [1 - z - e^{-z}]$ and for which the height range occupies a fairly large number, h, of units of z, the value of β is approximately 0.159h. Thus for a satellite at a height of 1000 kilometers and a typical scale height unit of 67 kilometers, the value of β is approximately 2.5.

In general, for a given experimental geometry, it should be possible to estimate the value of β a priori to an accuracy of about ± 10%.

The terms in the perturbation factors in Equations (42) and (43) have been left separate because of the way in which they have arisen. The terms for which β is a factor come directly from the non-linear dependence of the refractive index μ on the electron density which is proportional to X. The terms having $(\beta - 1)$ as a factor are due to ray refraction effects as summarized earlier. For a completely uniform medium $\beta = 1$ and these terms vanish as would be expected. For cases where β is very large, as might occur for example when the source is at a great height, $\overline{X}$ will usually become small and some alternative form of the equations may be desirable. We note that

$$\begin{aligned} \beta\overline{X} &= \overline{X^2} / \overline{X} \\ &\propto \int N^2 \, dh / \int N \, dh \end{aligned} \tag{44}$$

so that $\beta\overline{X}$ reaches a limiting value for a given ionosphere as the height range increases beyond the limit of the ionization distribution.

5.2 The Geometrical Parameter G

The parameter G is purely geometrical and involves the relative directions of the straight line, the magnetic field and the vertical at the ionosphere point. Since many of these directions are required in the computation of the quantity $\overline{B_L \sec \chi}$ used in first order analyses, it is a simple matter to compute G at the same time.

The form of G is interesting also.

From Equation (35)

$$G = \tan \theta [\tan \theta - \frac{Y_1}{Y_L}]$$

As noted earlier the field component normal to the plane of incidence does not enter the equation. Further, $G = 0$ at the zenith as would be expected. More surprising, however, is the disappearance of G when $Y_1 = Y_L \tan \theta$, i.e., when the field component in the plane of incidence is vertical. Under this condition, the second-order effects of wavenormal refraction and ray-wavenormal separation cancel each other leaving only the effect of the nonlinearity of the refractive index. This condition will occur at all latitudes when the plane of incidence is normal to the magnetic meridian plane, i.e., along directions which are approximately east and west of the observing station . Of course, G will be virtually zero everywhere for a station close to the magnetic pole.

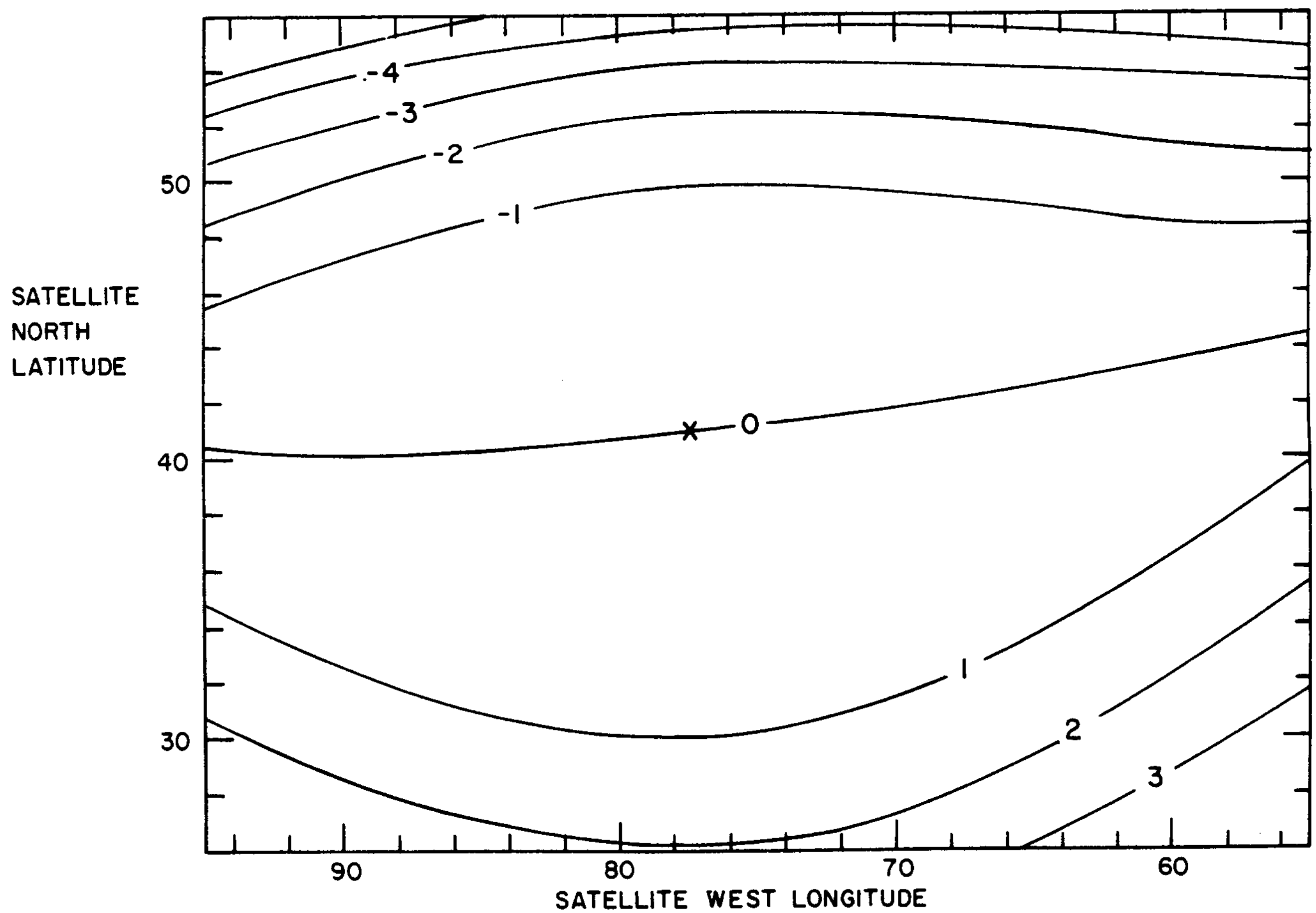
The disappearance of G for some azimuthal directions from the observing station has the interesting corollary that in these directions at least, the first order theory may well be better than one which attempts a second order analysis by only partially including the second order effects, for example by integrating the differential refractive index over the no-field ray trajectory. A similar situation may arise over a considerable field of directions for stations at high magnetic latitudes.

Maps of G have been prepared for two stations and are shown as contour maps in Figures 2 and 3. Figure 2 was compiled for an observing station at State College, Pennsylvania (40.8 N, 77.9 W), which is a typical mid-latitude station with a magnetic inclination of 72°. The contour G = 0 passes through the station normal to the magnetic meridian as discussed earlier, and G increases fairly uniformly in magnitude away from this line, positively to the north and negatively to the south. Some concentration of contour lines is seen far to the north, for which direction the magnetic field is more nearly transverse. The map covers the most probably useful range of source locations and corresponds to zenith angles at the ionosphere point of about 65 - 70° near the edges.

Figure 3 shows similar data for Huancayo, Peru (12.05 S, 75.35 W) which is almost on the dip equator. Here the uniform change of G with latitude comparable to that in Figure 2 takes place near the zero contour which in turn lies close to the locus of transverse conditions at the ionosphere point. Elsewhere G is positive and generally increases with distance from the observing station, with a magnitude comparable to the values found at mid-latitudes.

5.3 Accuracy of the Second-Order Equations

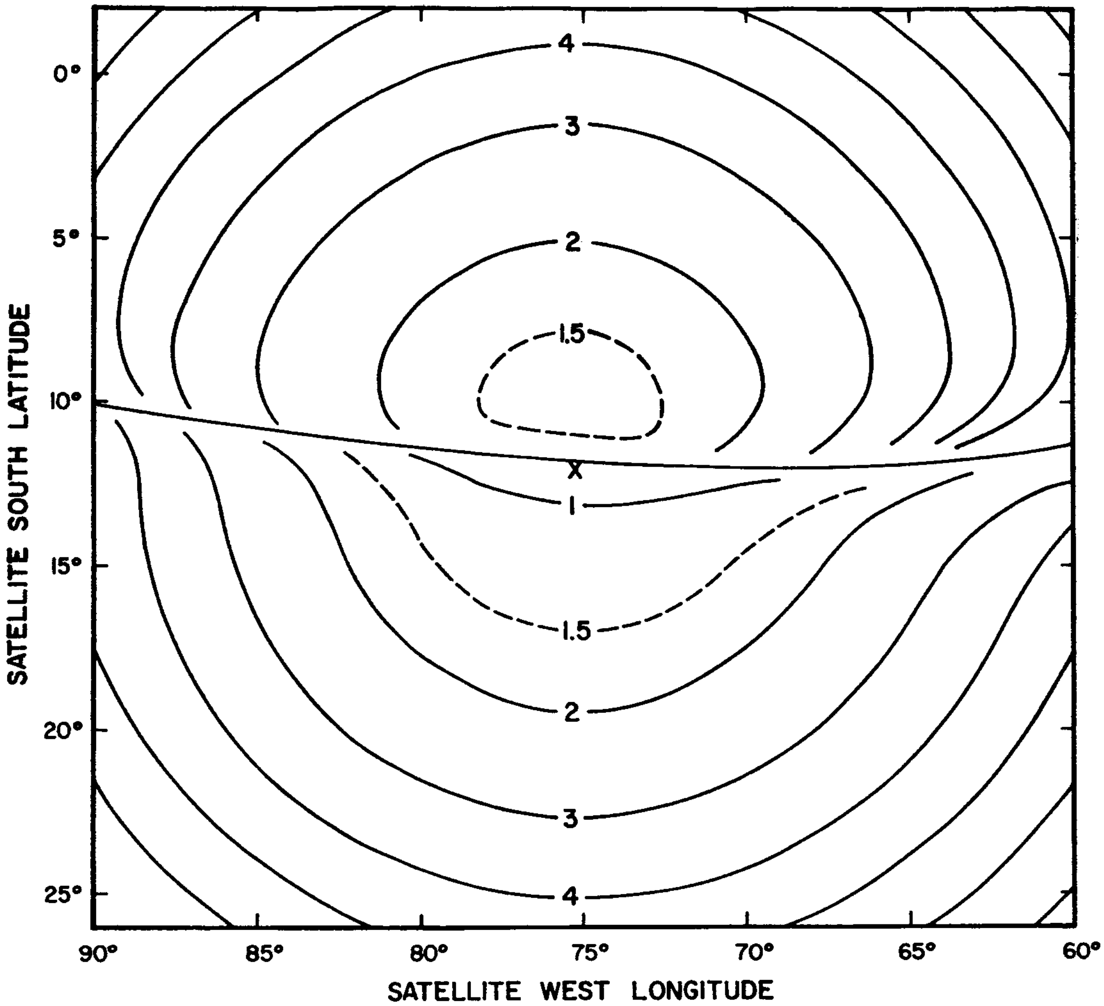
In order to test the adequacy of the second-order equations for the reduction of propagation data, comparisons were made with accurate numerical solutions of the propagation equations using particular ionospheric models. In all cases the same plane stratified ionization distribution was



CONTOURS OF THE SECOND-ORDER FACTOR G FOR STATE COLLEGE

(SATELLITE HEIGHT 1000KM, IONOSPHERE HEIGHT 300 KM)

FIGURE 2



CONTOUR MAP OF THE PARAMETER G

FOR HUANCAYO ,PERU X

SATELLITE HEIGHT 1000 KM ,IONOSPHERE HEIGHT 400 KM

FIGURE 3

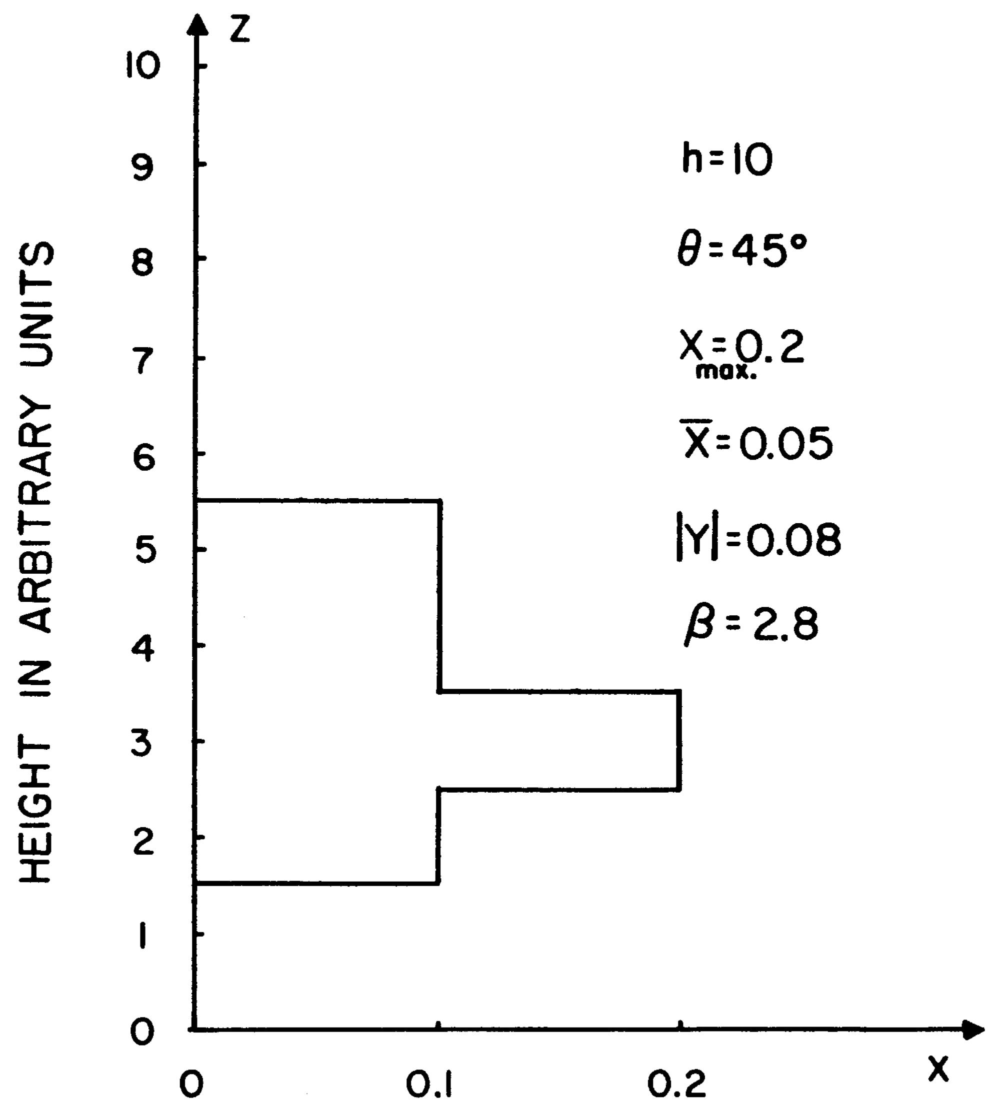
used, and ray paths for each QL mode were calculated by an iterative procedure similar to that of Lawrence and Posakony (1961) for a source at a zenith angle of 45^{o}, and for various directions of an assumed uniform magnetic field. The principal properties of the model are shown in Figure 4. With $X_{max} = 0.2$ and $\beta = 2.8$, this model is comparable to a daytime ionosphere under average sunspot conditions using a 20 mc/s satellite beacon at 1000 km. altitude. The value $Y = 0.08$ is also roughly typical of 20 mc/s operation.

The results of the computations for polarization rotation are shown in Table 1 for six different magnetic field directions which are indicated by their direction cosines. The percentage errors which resulted from use of the first order theory and of the second order equations (Equation (42)) are tabulated for comparison.

Table 1

Comparison of First-Order and Second-Order Interpretations of Polarization Rotation in Model Ionospheres. ($\beta = 2.8$, $\overline{X} = 0.05$, $Y = 0.08$). Source at $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

Model	Field	G	Percentage Error First Order	Percentage Error Second Order
A	0, 0, 1	0	+8.2	+0.6
B	1, 0, 0	2	+20.3	+0.9
C	$-\frac{1}{2}, 0, \sqrt{3}/2$	-2.732	-8.2	-3.5
D	$\frac{1}{2}, 0, \sqrt{3}/2$	0.732	+13.0	+1.2
E	$0, 1/\sqrt{2}, 1/\sqrt{2}$	0	+8.1	+0.5
F	$1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$	1	+14.4	+1.1



THE PARAMETERS
OF
TEST MODELS

FIGURE 4

Models A and E are respectively for a vertical field and for a field whose component in the plane of incidence is vertical. As discussed earlier these two cases contain similar errors in first order analysis and are about equally improved by the use of the second order equations. The 8% error in the first order analysis arises almost entirely from the nonlinearity of the refractive indices for these two models.

Models D and F correspond to rays almost parallel to the field or its component in the plane of incidence and the errors are again very similar whether there is transverse field component or not.

Model B, which shows the largest error of the set in first order analysis, is for a horizontal field, showing the importance of higher order analysis for low latitude stations where the several refraction effects are additive.

Model C which shows the least improvement with second order analysis is for a case where propagation is within 15° of transverse to the magnetic field. In such cases the procedure of rounding off the propagation equations to order three does not ensure greatly improved accuracy because the transverse field component is much larger than the longitudinal component which appears in the leading term of the polarization rotation equations.

The considerable improvement in accuracy shown by the second order analysis in Table 1 may not always be realized in practice. In some cases shown the improvement is fortuitously high, and also it must be remembered that the cases considered are for plane stratified models with a uniform magnetic field, and that the value of β is known exactly. Practical use of the equations would involve some uncertainty in the value of β, while the effects of sphericity and departures from horizontal stratification might also be expected to degrade the results to some extent. However, there is no doubt that a substantial improvement in accuracy can be realized, perhaps of an order of magnitude for typical cases.

Table 2 shows a tabulation of phase path comparisons made using the same models as before.

Table 2

Comparison of First-Order and Second-Order Interpretations of Phase Path Reduction in Model Ionospheres

Model	Field	Y_L	Upper sign First-Order	Upper sign Second-Order	Lower sign First-Order	Lower sign Second-Order
A	0,0,1	0.05656856	+0.8%	+0.6%	+13.0%	-0.7%
B	1,0,0	0.05656856	+0.2%	+0.2%	+13.8%	-0.1%
C	$-\frac{1}{2}, 0, \sqrt{3}/2$	0.0207055	+4.7%	+0.5%	+8.5%	-0.5%
D	$\frac{1}{2}, 0, \sqrt{3}/2$	0.0772741	-1.5%	+0.6%	+16.0%	-0.7%
E	$0, 1/\sqrt{2}, 1/\sqrt{2}$	0.0400	+2.4%	+0.5%	+11.0%	-0.5%
F	$1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$	0.0653197	-0.4%	+0.4%	+14.5%	-0.5%

It is seen that the use of the second order equations results in an accurate value for the phase path reduction in all cases.

Probably the most striking feature of Table 2 is the relatively high accuracy seen in the first order analysis for the upper sign. This comes about largely from the accidental similarity between the values for $\overline{X}$ (0.05) and Y_L for the models chosen, and is evidently less pronounced when Y_L differs markedly from 0.05 in value as for example in Models C and E. The particular value of β also plays an important role in this result. A much less pronounced disparity in the accuracies of first order analysis for upper and lower signs would be expected for example in a nighttime ionospheric model, or for one appropriate to higher frequencies, for either of which the value of $\overline{X}$ would be much smaller than here.

Nevertheless, it is evident from the form of the second order equation (Equation (43)), that the second order correction factor will be less whenever the value of Y_L is positive.

Thus it will usually be preferable to perform phase path reduction measurements using the magneto-ionic mode corresponding to the positive sign for Y_L, i.e., using North into West circular polarization for a mid-latitude station in the Northern hemisphere. In this way the second order correction factor, which contains some uncertainty through the parameter β, is minimized and the probable accuracy of the final result is improved.

With the improved accuracy which results from the use of the second order equations it might seem desirable to extend the equations to still higher orders of accuracy. To do so, however, is to introduce greatly increased complexity into the equations, including more detailed ionization distribution functions and field geometry parameters, and the azimuthal deviation effects noted earlier. It must be remembered that all these equations are derived in terms of a simplified model whose fit to a real ionosphere is only approximate, so that in practice the accuracy of the results is limited. It is proposed that if higher accuracy analyses are to be made, these are best done by means of a full computer ray tracing procedure such as that described by Lawrence and Posakony (1961), which may include such effects as magnetic field variation with position, departures from horizontal stratification, etc.

In applying the second-order equations, it must be remembered that they have been derived on the assumption of quasilongitudinal propagation, and they should be used with increasing caution if the experimental conditions should approach the limits of this assumption. An indication of their falling accuracy was seen in Table 1 as more nearly transverse conditions were approached.

6. Application to Propagation Experiments

6.1 Polarization Rotation

There are a number of experiments in which the rotation of the plane of polarization of a radio wave passing through the ionosphere can be measured unambiguously, and for which the inclusion of the second order corrections is a straightforward matter through the use of Equation (42).

For example, measurements between a ground station and an ascending rocket enables the rotation at any time to be found by counting from the time of launch. In such an experiment the values of β, $\overline{X}$ and G change progressively with time and a modified form of the equations may be preferable to that given here. For satellite transmitters the ambiguity may be resolved in cases where the propagation at some time is transverse to the magnetic field, at which point the rotation is essentially zero, and from which point the rotation may be counted e.g., Blumle and Ross (1962). In some experiments the wave frequency is so high that the total rotation may be small enough to be unambiguous, e.g., Millman et al (1960). Usually in such cases, however, the correction given by second order analysis is so small as to be insignificant.

Recently, some measurements have been proposed of the polarization rotation in incoherent backscatter from the ionosphere, where, as in the rocket case, rotation may be found by counting from the bottom of the layer. In these experiments also a modified form of the second order equations may be used to give improved accuracy.

6.2 Polarization Rotation Rate

In many propagation experiments, particularly those involving satellite beacon transmitters, it is not possible to measure the polarization rotation angle directly but only changes in rotation with time. In these measurements the changes usually result from a changing experimental geometry rather than from an ionospheric medium which changes explicitly with time. To interpret such measurements it is necessary either to assume a stratified medium or to assume some form of its variation along the locus of the ionospheric point.

If the horizontal variation of the ionosphere can be included adequately in a first order theory, then it will be worthwhile to consider including the second order refraction corrections as an additional refinement. It will usually be most convenient to do so by converting the time dependence of the rotation angle to a position dependence, so that the appropriate value of the parameter G may be associated with each point on the record. The second order equations will be slightly in error if the ionosphere is not horizontally stratified, but this effect is expected to be small and will be neglected here.

Denoting satellite positions by letter subscripts we have from Equation (36)

$$\Omega_i = \Omega_{io} \left[1 + \frac{1}{2} \beta_i \overline{X}_i + \frac{1}{2} (\beta_i - 1) G_i \overline{X}_i \right] \qquad (46)$$

$$\therefore \Delta\Omega = \Omega_i - \Omega_j = (\Omega_{io} - \Omega_{jo}) + \frac{1}{2}\,\Omega_{io}\,\overline{X}_i\,[\beta_i + (\beta_i - 1)\,G_i]$$

$$- \frac{1}{2}\,\Omega_{jo} - \overline{X}_j\,[\beta_j + (\beta_j - 1)\,G_j] \qquad (47)$$

The second-order correction to the differential rotation angle in Equation (47) will usually be fairly small and may be calculated if values derived from first order theory are inserted and the presumed known form of the horizontal variation of electron content is included.

The measured differential rotation angle may then be corrected to an equivalent value suitable for first order analysis. If we suppose that $\overline{X}_j = \alpha \overline{X}_i$ then

$$\Omega_{io} - \Omega_{jo} = K\,\overline{X}_i\,[Y_{Li}\sec\theta_i - \alpha Y_{Lj}\sec\theta_j] \qquad (48)$$

whence $\overline{X}_1$ may be found.

Inspection of the field map for G for a mid-latitude station (Figure 2) shows that since G changes uniformly from north to south the second order correction factor varies similarly, so that first order analysis of polarization data from a source which moves across the field of view will indicate the presence of a spurious gradient in electron content. Similarly the use of first order equations in polarization rotation rate experiments will lead to values of electron content which are too high for a north-going satellite source, and too low for a south-going source, even for a horizontally stratified ionosphere. A decrease in electron content from south to north will, of course, produce a similar effect in the rotation rate experiment.

6.3 Close-Spaced Frequency Polarization Rotation Dispersion

One way in which the ambiguity in polarization rotation angle can be resolved, is by the simultaneous use of two closely spaced frequencies between which the differential rotation angle is unambiguously small. This technique has been used in moon echo experiments, e.g., Evans (1957), and is proposed for some satellite beacon experiments, notably S-66 and OGO-A.

On first order theory, (Equation (1)) the rotation angle varies inversely as the square of the frequency so that a measured dispersion is readily transformed into a total rotation angle.

For if

$$\Omega \propto \frac{1}{f^2}$$

$$\frac{d\Omega}{\Omega} = -2 \frac{df}{f} \qquad (49)$$

When second-order effects are included, however, it is seen that the rotation angle no longer follows a simple inverse frequency squared law but contains terms in $\overline{X}^2$ which vary as the inverse fourth power of frequency and whose dispersion is greater than that of the leading term.

$\therefore$ If

$$\Omega = A \overline{X} \left[1 + \frac{1}{2} \beta \overline{X} + \frac{1}{2} (\beta - 1) G \overline{X} \right] \qquad (50)$$

then

$$\frac{d\Omega}{df} = A \frac{d\overline{X}}{df} \left[1 + \beta \overline{X} + (\beta - 1) G \overline{X} \right] \qquad (51)$$

$$\therefore d\,\Omega = -2\overline{X} \;\frac{df}{f}\; [\,1 + \beta\overline{X} + (\beta - 1)\,G\,\overline{X}\,] \qquad (52)$$

Thus the second order correction terms in the rotation dispersion are twice as great as for the polarization rotation itself, and the inclusion of this correction is correspondingly more important.

The proposed polarization dispersion experiments using satellite transmitters will use frequencies of 40 mc/s and 41 mc/s. Adapting the first order analyses of Table 1 to a realistic daytime ionospheric model in this frequency range will reduce the errors in first order analysis shown in the Table by a factor of four approximately. Thus the use of first order equations for reducing the 40-41 mc/s differential polarization data may be expected to lead to errors ranging from 4% to 10% for the models in this Table.

6.4 Wide-Spaced Frequency Polarization Rotation Dispersion

One polarization rotation experiment of particular interest is that involving the relative rotation rates at wide-spaced frequencies, usually in harmonic ratio. Using first order theory one would expect that the rotation angles and rotation rates would be in the inverse ratio of the squared frequencies. Departures from this value may be used to infer some properties of the ionosphere. Yeh (1960) has considered this effect, although his analysis did not include the second-order refractive effects fully.

For wave frequencies f_1 and $n\,f_1$ the respective rotation angles are

$$\Omega_1 = \Omega_{01}\;[\,1 + \tfrac{1}{2}\,\beta\,\overline{X}_1 + \tfrac{1}{2}\,(\beta - 1)\,G\,\overline{X}_1\,] \qquad (53)$$

and $$\Omega_n = \Omega_{0n} \left[1 + \frac{1}{2}\, \beta\, \overline{X}_n + \frac{1}{2}\, (\beta - 1)\, G\, \overline{X}_n \right]$$

$$= \frac{1}{n^2}\, \Omega_{01} \left[1 + \frac{1}{2n^2}\, \beta\, \overline{X}_1 + \frac{1}{2n^2}\, (\beta - 1)\, G\, \overline{X}_1 \right] \tag{54}$$

$$\therefore\ n^4\, \Omega_n - \Omega_1 = (n^2 - 1)\, \Omega_{01}$$

$$\therefore\ \Omega_{01} = \frac{n^4 \Omega_n - \Omega_1}{n^2 - 1} \tag{55}$$

Thus a value for Ω_{01} and hence for $\overline{X}_1$ may be found without the need for an estimate of the value of β or of G.

Also from Equations (53) or (54) we may calculate the value of β by substituting the value of Ω_{01} from Equation (55) and the value of G for the particular position.

As indicated in Section 6.2, the polarization rotation angle is often not known explicitly but only its changes can be measured. In this case we may follow a similar procedure by evaluating Equation (47) for each of the two frequencies and combine them, whence

$$n^4\, \Delta\Omega_n - \Delta\Omega_1 = (n^2 - 1)\, \Delta\Omega_{01} \tag{56}$$

Thus a differential rotation angle $\Delta\Omega_{01}$ which would be produced if propagation followed first order theory, may be found from the measured differential rotation angles $\Delta\Omega_1$ and $\Delta\Omega_n$, without the necessity for assuming any particular horizontal dependence of the ionosphere. The interpretation of this differential angle, however, using Equation (48), will require the assumption of this dependence.

6.5 Phase Path Dispersion and Doppler Dispersion

Since the phase path reduction cannot usually be measured directly, comparison methods become essential for the interpretation of experimental data. The normal observation is of the Doppler frequency shift which is proportional to the rate of change of phase path with time. This can be converted to a corresponding rate of change of phase path reduction if the free space Doppler effect can be removed, for instance by accurate knowledge of the experimental geometry, or more commonly by harmonic frequency dispersion measurements.

The ionospheric contribution to the Doppler frequency shift may be found by differentiating Equation (40) with respect to time. Assuming a horizontally stratified ionosphere, this leads to an ionospheric Doppler shift of

$$\Delta f_D = \frac{1}{\lambda}\frac{d}{dt}(\Delta P) = \Delta f_{DO}\left[1 + \frac{1}{4}\overline{X} + \frac{3}{4}(\beta - 1)\overline{X}\sec^2\theta\right] \mp \frac{1}{2}\frac{h\overline{X}}{\lambda}\frac{d}{dt}(Y_L \sec\theta) \tag{57}$$

where Δf_{DO} is the ionospheric Doppler shift calculated by first order theory.

The last term is simply one half the rate of occurrence of polarization fades given by first-order theory, and has been shown by Bowhill (1958) to be constant for the case of a satellite source moving horizontally with respect to a plane stratified ionosphere with a uniform magnetic field.

Differentiating Equation (57) again and evaluating the result when the zenith angle is a minimum we find

$$\frac{d}{dt}(\Delta f_D)_{\theta min} = \frac{d}{dt}(\Delta f_{DO})\left[1 + \frac{1}{4}\bar{X} + \frac{3}{4}(\beta - 1)\bar{X}\sec^2\theta_{min}\right] \tag{58}$$

$$\therefore \quad \bar{X} = \frac{\ddot{\Delta P}}{P}\left[1 - \frac{1}{4}\bar{X} - \frac{3}{4}(\beta - 1)\bar{X} - \frac{3}{4}(\beta - 1)\bar{X}\tan^2\theta_{min}\right] \tag{59}$$

which is closely similar to the semi-empirical equation used by Ross (1960) to correct first order theory for the effects of refraction.

A discussion of the effects of a non-stratified ionosphere on these equations will not be entered into here.

As was the case with the polarization rotation effect (Section 6.3) the use of closely spaced frequencies may be used to provide an instantaneous comparison measurement, rather than using the time dependence of phase path with its attendant assumptions of the horizontal distribution of ionization, e.g., Eshleman et al (1960). These experiments are equivalent to finding the group delay in the composite signal from the phase dispersion. From Equation (40) the ionospheric contribution to the group delay, τ, is given by

$$\tau = -\frac{d}{df}\left(\frac{\Delta P}{\lambda}\right) = \tau_o\left[1 \mp 2Y_L + \frac{3}{4}\beta\bar{X} + \frac{3}{4}(\beta - 1)\bar{X}\tan^2\theta\right] \tag{60}$$

where $\tau_o = \frac{-d}{df}\left(\frac{\Delta P_o}{f}\right) = \frac{1}{2}\,\frac{h\bar{X}\sec\theta}{c}$

Hence $\overline{X}$ may be found by inversion of Equation (60)

It is important to note the increased magnitude of the second-order correction factor, compared with its value in the original phase path formulae. The magnetic field effect has been doubled while the refraction term and non-linear index term are each tripled in importance, so that, referring to the test computations in Table 2, it is seen that the first order analysis may be in serious error. Using the same model parameters as in Table 2 the following Table results.

Table 3

Errors Arising From use of First-Order Analysis of Phase Dispersion in Close Spaced Frequencies for Several Test Models ($\overline{X}$ = 0.05, β = 2.8)

Model	Upper Sign	Lower Sign
A	+5.9%	+28.6%
B	+5.9%	+28.6%
C	+13.1%	+21.4%
D	+1.8%	+32.7%
E	+9.3%	+25.3%
F	+4.2%	+30.3%

Although experiments of this sort would probably not be performed using such a low operating frequency as these data represent, nevertheless the errors are such as to merit serious attention even for much higher frequencies.

6.6 Hybrid Faraday-Doppler Experiments

It has been shown by de Mendonca and Garriott (1962) that by the use of simultaneous measuremements of Doppler dispersion and polarization rotation, the electron content of the ionosphere can be determined without the assumption of a particular form of the horizontal distribution of ionization. This method was based on the first order propagation equations for each effect.

The extension of de Mendonca's formulae to include the second order effects discussed here leads to a very cumbersome formulation and will not be made here. The most straightforward procedure for the inclusion of these effects appears to be to make the first order analysis of the data to arrive at approximate values for the total polarization and phase path reduction, to correct these initial values to equivalent first order values by the methods outlined here, and iterate the solution to arrive at improved values.

6.7 Experiments Using Distant Sources

When the source is at a very great height, some simplification of the basic second order equations (Equations (36) and (40)) becomes possible. As was shown in Section 5.1, under these conditions β becomes very large, so that we may write for Equation (36)

$$\Omega = \Omega_o [1 + \frac{1}{2} \beta \bar{X} (G + 1)] \qquad (61)$$

and for Equation (40)

$$\Delta P = \Delta P_o [1 \mp Y_L + \frac{1}{4} \beta \bar{X} \sec^2 \theta] \qquad (62)$$

It should be pointed out, however, that in such experiments, if the ionization is distributed in appreciable amounts over a substantial part of the height range, the equations of this paper must be used with caution, because (1) the assumption of a uniform field becomes less realistic and (2) the use of plane stratification of ionization does not give a good representation for the higher ionization.

Within these limitations, however, the above Equations (61) and (62) may still be expected to give improved results over those of the first-order theory for moon radar and distant satellite beacon experiments using either Doppler or polarization rotation effects.

7. Summary and Conclusions

Methods have been developed for the analytical determination of the effects of refraction and nonlinearity of the refractive index for a quasilongitudinal magneto-ionic medium through which high frequency radio waves are propagated. The results are still approximate, but they extend the accuracy of the commonly used first order analysis methods by about an order of magnitude.

It has been shown that it is possible to estimate some of the second-order coefficients approximately from a priori knowledge of the general form of the ionization distribution, while others can be calculated very simply from the known experimental geometry. In all cases emphasis has been placed on casting the equations in an analytical form suitable for manual data reduction, avoiding the need for computer ray-tracing programs which have characterized earlier inclusion of these effects. As a consequence it is possible to determine by inspection the effects of each of the several parameters which enter the equations, and to assess in advance the sensitivity of the result to the tolerances in these parameters. Some general conclusions concerning the choice of magneto-ionic mode in doppler studies and the geomagnetic dependence of the second-order polarization effects, for example, have been drawn from the equations.

Studies have been made of the effect of the more accurate equations in the analysis of data from a number of types of propagation experiments. In some cases, notably those involving dispersion in the propagation effects, these studies have indicated that surprisingly large errors may result from the use of the simple first order analysis methods.

It is strongly urged that, when magnetic field maps are computed for first order polarization rotation analysis, the second-order parameter G be computed as well so that the analysis may be upgraded by means of the second-order equations.

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